AN ANALYSIS OF STUDENTS' MATHEMATICAL ERRORS IN THE TEACHING-RESEARCH PROCESS

Maria Legutko
Pedagogical University of Kraków (Poland)
maria.legutko@gmail.com

ABSTRACT
This study attempts to analyze students' mathematical errors in the teaching-research process from teachers' perspective. Students' errors are inevitable in learning mathematics; they ensue from mathematics itself or are results of teaching. Teachers cannot be afraid of errors, but should create such situations in which students reveal their errors so that teachers are able to methodologically correct them. Two examples of students' errors are described in this article, together with their hypothetical causes, suggested corrective methods – actions that teachers may take in a given situation.

1. INTRODUCTION

In the teaching-research process teachers have to thoroughly analyze students' errors, attempt to understand the errors, explain what they consist in, and find what causes them. Depending on the conclusions of such an analysis, teachers should select corrective means and methods in order to deepen their students' understanding of mathematical concepts, improve their reasoning methods and to perfect their skills. In order to achieve that teachers need certain knowledge about errors and the methods of response to errors.

2. ERRORS IN LEARNING MATHEMATICS ARE INEVITABLE

2.1 Mathematics creates an internally coherent structure and some concepts are built on the basis of other concepts, therefore learning mathematics is difficult, requires regularity and (self) control. A seemingly small gap in comprehension or knowledge creates further misapprehensions that are built one upon another, and which after some time are revealed in an error avalanche. An unrevealed error, which is rooted in the mind of students, is therefore a major threat to the construction of students' mathematical knowledge. A revealed and clarified error may be extremely useful both for students and teachers (Krygowska, 1988: Booker, 1988).

Mathematics is an abstract science and uses a specific language. Since the onset of mathematics education we try to create abstract concepts of natural numbers, operations, and geometric figures in children's minds. Students who begin such education live and function in the real world, often think and act in a concrete way, when the first mathematizations of real problem situations as well as the first interpretations of abstract mathematical objects in reality take place at this first educational stage. This is when numerous difficulties and linguistic errors appear together with a simultaneous attempt at precision in mathematical language and the usage of a natural language that is comprehensible for children. Also, first changes of the meaning occur – of the terms taken from reality into mathematics (e.g. circle and perimeter; area). This first transition from the real world to the world of mathematics is extremely important for the entire mathematics education and the establishment of the conceptions on what mathematics is and what its role in life and education is (Krygowska 1988, 1977; Pellerey, 1988). "To understand how errors emerge and how they might be overcome on the simplest level of fundamental procedures is priceless for teachers and students, since it enables them to master the procedures of a higher level" (Booker, 1988).

Various contradictions are rooted in mathematics itself. Mathematics education requires overcoming these contradictions. The common juxtapositions are: (i) attempts at algorithmization versus creative and conscious actions; (ii) natural thinking of every day life vs. formal reasoning based on accepted conventions (e.g. veracity of an alternative or implication); (iii) abstraction of mathematics as a science vs. connections of mathematics with the real world; (iv) what was considered true vs. can turn out to be an error (for example the famous Cauchy Theorem on boundary continuity of converging sequence of continuous functions); (v) a rejected error vs. can become a source of development of new knowledge. These and other contradictions create numerous misapprehensions and might be the reason for numerous students' (and teachers') errors (Krygowska, 1988a; Rouche, 1988; Pellerey, 1988).

What seems simple and obvious in mathematics does not necessarily have to be simple and obvious in mathematics teaching. It was noted among others by Freudenthal (1988) and Thom (1974) who analyzed the 'new math' imposed on educational systems in the 1960s that focused on the concepts of the set and function. Educational systems: mathematical-philosophical-pedagogical concepts, curricula and text books also have an impact on many of students' (and teachers') errors.

2.2 Role errors play in teaching and learning mathematics

39th Meeting of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) in 1987 in Sherbrooke, Canada, was devoted to the role errors play in teaching and learning mathematics. It was attempted to perceive errors that appear in mathematics teaching from philosophical, moral, psychological, mathematical and didactic perspectives. While seeking to define an error it was agreed that an *error takes place when a person chooses the false as the truth*. When the actual result does not correspond with the assumed aim (we talk then about an *erroneous result*); when the undertaken actions are not compatible with the accepted procedures (we talk about *erroneous actions*). There are also *erroneous conceptions* (approaches), which might significantly hinder the problem solving and generate irrational actions of a person.

It is impossible to create a sensible error typology, since the error diversity is as rich as human life. The same error might also be analyzed from very different points of view. The analysis of error causes and the application of these analyses in the process of mathematics teaching planning and learning are more interesting for the teaching process than as an error classification. When introducing new concepts or procedures, knowledge about errors informs teachers what to focus on, what to clarify, how to negotiate the comprehension of new terms in order to avoid a certain type of errors, and how to positively use the occurring errors. However, also in this approach to errors there is a need for some terminological negotiations, a distinction between mistakes and *"essential" errors* and *erroneous conceptions* deeply rooted in the mind of students (Krygowska, 1988a). These erroneous conceptions are associated with epistemological obstacles (Brousseau) and with pragmatic errors (Pellerey).

A *mistake* is a result of the lack of concentration control or weak memory. We make a mistake when we incorrectly apply a formula or theorem, which we know (or should know) from theory acquired earlier.

An *error* reveals inadequacy of knowledge and is closely connected with imagination and creativity in a new situation, and is caused by an insufficient mastery of basic facts, concepts and skills. Such an error in learning something new is called *normal* by Duverney (quoted in Rouche, 1989).

2.3 Mathematical errors and didactic errors

In teaching practice and mathematics lesson analysis we use the terms *mathematical errors* and *didactic errors*. A mathematical error is made by a person (student, teacher) who in a given moment considers as true an untrue mathematical sentence or considers an untrue sentence as mathematically true. Didactic errors refer to a situation when teachers' behavior is contradictory to the didactic, methodological and common sense guidelines.

Mathematical errors are discussed: (i) *in defining mathematical concepts and application of definitions* (omission of essential characteristics in a given class of objects, or inclusion of inessential characteristics into the definition), examples: "a cube is a solid which has six congruent faces;" "parallelogram is a quadrilateral with opposite sides parallel and of equal length;" (ii) *in theorem understanding and application* (using the hypothesis without testing the proposition), examples include: "the sum of indefinite geometric sequence 2,4,8,16,.. with a term a_1 =2 and q=2 equals -1 because $s=a_1/(1-q)$;" (iii) *in mathematical method*, examples: too quick, unjustified generalizations made on the basis of observing a few particular cases; justifying the theorems referring to any triangle for equilateral triangle; (iv) *in algebraic expression and formula transformations*, examples include (degenerated formalism): $2a - a = 2$; $x \cdot x = 2x$; $(a+b)^2=a^2 + b^2$; and (v) *in using mathematical language*, examples include: interchangeable usage of terms: "digit" and "number;" understanding the term "any number" as a concrete number, which you can freely choose; "cube's side" instead of "edge;" "odd function is a function that is not even," and errors associated with figure comprehension in geometry.

Teachers' didactic errors associated with teaching a class: (i) *incoherent structure of teaching content*, examples: division of two decimal numbers is not preceded by division and multiplication of a decimal number by any power of 10; solving equation sets of the first degree with two variables is not preceded by an equation of the first degree, with two variables and its interpretation on the coordinates on the plane; (ii) *unsuitable selection of examples used in forming a concept*, examples: only one height of parallelogram is discussed (between the longer sides); only one height is discussed in an acute-angle triangle – the one that is parallel to the side of a sheet of paper; (iii) *unsuitable selection of problems used for aim realization*, examples: only simple problems and tasks are solved, and there are no problematic problems; too difficult examples are chosen to illustrate a method; (iv) *underestimating the necessity to master the basic skills* by students, such as correct calculations, representation and comprehension of the data on graphic representation of geometric figures; (v) *inappropriate teachers' response to students' error*, e.g. irritation due to numerous recurrences of errors that have been clarified in class; (vi) *inaccurate selection of methods for subject realization*, e.g. teachers employ a presentation method, when students could discover some theorems themselves by solving accurately selected problems; (vii) *class work is based on the activity of selected students* (the best or average ones), e.g. when the same students solve problems all the time; those who are able to, while the rest of the class copy the solutions from the board thus reinforcing their belief that they themselves are not able to solve the problems, or are bored because they are able to solve such problems easily.

3. TEACHERS SHOULD UNDERSTAND STUDENTS' ERRORS, CONTEMPLATE THEIR CAUSES AND METHODOLOGICALLY CORRECT THEM

3.1 Teachers' response to students' errors

Teachers are afraid of students' errors. Teachers' fear of students' errors is often manifested by teachers asking students many questions in order to navigate them to a correct answer and to avoid an error. A revealed error requires explanation and its correction takes time. Beginning teachers are often afraid that they might be surprised by students' errors and that they might at first overlook them or not to know how to respond to them. It is useful then for teachers to speculate about possible students' errors while preparing the lesson ("predicted errors").

Students' errors influence the grading process and result in a lower evaluation of students' knowledge or skills. A brilliant idea how to solve a problem followed by errors in its implementation (erroneous actions) cannot be graded high. (The best example of such an approach is a common comment among teachers "Fantastic, but you only get a C").

Experienced teachers might be annoyed by persistently recurring students' errors. Teachers have to be prepared to explain something "not 7, but 77 times" in numerous ways, before the students' reasoning and actions change according to expectations. It is especially true for errors ensuing from students' previous knowledge, which has been sufficient to a certain degree but which fails in some situations (epistemological obstacles and erroneous concepts), as well as errors resulting from erroneous conceptions. Even if students correct such errors themselves a change in their conception does not necessarily occur and after some time they will make the same errors again.

Teachers are afraid of students' errors also because the errors might inadvertently result from their teaching. Students' errors might reveal lack of skill or comprehension which should have been mastered earlier on in the teaching process. Freudenthal (1989, 109) believes that "students who make errors always do so with the teacher who teaches them; at least partially the error's role is connected with the teachers' role in the learning process." Similarly, according to Booker (1989, 101) "the origins of many errors are rooted not so much in students but in the manner children are introduced to mathematics."

3.2 Teachers' strategies to deal with students' errors

Teachers have to *accept students' right to err*, especially when students face a new, unusual situation. A familiar action scheme cannot be immediately applied in an unusual situation, it has to be either adjusted (accommodation) or a new scheme has to be formed in order to solve a problem.

Teachers should try to *understand students' errors*, try to understand the way students think, because "children do not make errors in mathematics thoughtlessly; they either believe that what they are doing is correct, or are not at all sure what they are doing" (Booker, 1989, 99). It is then worthwhile to often ask students a question: *"why* *so?"* If the errors result from inattentiveness then students will correct them quickly; if students are not sure that what they wrote is correct then they will quickly erase or cross it out; if students are sure they are right they will defend their opinion, often with determination. If the question is asked not only when students make errors, but also when everything is correct, it forces a moment of reflection and creates an opportunity to justify directly formulated conclusions. If despite the question, *students still do not recognize their errors,* teachers' encouragement e.g. "check your general reasoning in this concrete case," "give a counterexample," "estimate if it is possible," often leads to contradictions and might facilitate error recognition (Krygowska, 1977, 1988).

However, students do not always recognize their errors immediately, teachers then might use other students to correct and explain the mistakes. It is not only about having other students give the correct solution, but also about having them explain what was incorrect and why. It also teaches good students how to explain errors, what arguments to use in order to convince others that problems might be solved differently. Sometimes an error analysis needs to be postponed, and an immediate and efficient reaction is not always possible. It requires reflection, conversation and sometimes a piece of advice from more experienced teachers.

To err is human (*errare humanum est*) as the Latin saying has it; however, the important thing is what conclusions are drawn from errors, how we learn while erring (*errando discimus*). It is easier to overcome difficulties when they arise and are not solidified yet than to unlearn erroneous reasoning habits or skills, which have become incorporated into thinking and acting of a given person. Hence significant conclusions emerge for teaching mathematics: (1)– control (but do not evaluate) early enough during the first stage the comprehension and correctness of performed actions, in order to capture and correct normal errors occurring then; and (2) teach students how to control themselves if they make errors and how to correct them, to teach them how to "master errors."

A systematic self-control (auto-control) *teaching* consists in teaching students certain strategies, certain actions which increase their chances for a correct final solution. They do not guarantee that students will not make errors or that they will recognize errors in their solution and will be able to correct them, but they facilitate correctness control and error recognition (Polya, 1945; Krygowska, 1988; Ćwik, 1987; Turnau, 1990; Dybiec, 1996). Such strategies include: (S1) step by step control; (S2) solving problems by using various methods; (S3) checking a general solution in a given case; (S4) checking data specification; (S5) making a graphic representation; (S6) estimating the solution by an approximate calculation; (S7) estimating if the solution is sensible and if it is possible; and (S8) referring to the real context.

4. EXAMPLES OF ERROR ANALYSIS IN THE MATHEMATICS TEACHING-RESEARCH

A collection of errors analyzed from various points of view can be found in each of the above-mentioned works. I shall add two more examples to this collection. They took place during the work of the Rzeszów-Kraków PL1 teacher team in the project "Transforming Mathematics Education through Teaching-Research Methodology PL-Comenius-C21."

4.1. Example 1.

Thirteen-year-old students received the following problems to solve from their teacher:

1. The square area equals 9 cm^2. What is the length of the side of this square?
2. Solve the equation: $x^2=16$

Students gave only positive solutions in the first problem $a=3$ and in the second problem $x=4$. They were surprised that in the first problem (in which they most often drew a square) the answer was correct, while in the second problem it was not. They had never learned how to solve quadratic equations before. When asked about another solution of this equation, they were unable to give it (Migoń, 2007).

This error can be called a mathematical error – students considered as true a false statement – the solution to the equation $x^2=16$ is 4. Depending on what a reasoning manner the students used to achieve the result, they could make various errors along the way: in comprehending the concept "to solve an equation," in application of the theorem on root extraction or in comprehending the instruction "to solve an equation."

4.2 Example 1 – analysis

When analyzing this error we distinguished *hypothetical origins* (H) *of this error*:

(H1) the geometrical context influence of problem 1. Students become familiar with quadratic equations of this type first in geometric situations and have no awareness that such equations have two solutions, one positive and one negative in the algebraic context;

(H2) students know that when $x=4$ then $x^2 = 16$, so they use a reverse operation, write the symbol of equation they solve $x^2=16$ / $\sqrt{}$ and obtain $x=4$; they act analogically to the situation with the equation of the first degree marked with one variable and perform reverse operations;

(H3) when students were asked about other solutions to the equation, they did not look for another root in the set of negative numbers. They have much more experience in operations on positive numbers than on negative numbers;

(H4) students are happy that they found a solution and they have no awareness that the equation is solved only when all the numbers that meet the requirements of the equation are given;

(H5) students do not know what set might contain solutions to the equation of a given type, e.g., the solution to a linear equation can be one number, any real number or such a number might not exist (empty set); the solution to a quadratic equation can be one number, two numbers or such numbers might not exist in the number sets that we are familiar with.

Can we talk about the teacher's didactic error here? What did the teacher want to find out about his students (in a diagnostic test) when he put together these two problems? Students had not yet learned how to solve equations of the second degree but they were familiar with negative numbers, they were able to perform operations on negative numbers (they knew that the product of negative numbers is a positive number), they were familiar with the root extraction of natural numbers, and they knew what it meant to solve equations of the first degree with one variable. It might be said that they possessed sufficient knowledge. They only faced an unusual situation. The teacher's intention was to create an opportunity to discuss with the students – how the role of letters in mathematics depends on the situation context.

4.3 Example 1 – suggested corrective methods

How to teach equation solving:

(M1) constantly and consistently remind students what an equation and its solution are and what it means to solve an equation;

(M2) contrast the concept of equation $x+1 = 2$ with inequalities: $2+1 >2$ and $(-1)+1<2$, make students aware that the equality is true for x=1 because $1+1 =2$;

(M3) having introduced operations on whole negative numbers, solve riddles with students, e.g. "I'm thinking of a number, I multiplied the number by the same number and as a result I obtained 16. What is the number I have in mind?" and search for positive and negative solutions;

(M4) write down the solutions of quadratic equations in the problems with geometric context in the following manner: (a) when calculating the length of the side of the square $a^2= 9$ $a= -3$ or $a=3$ but $a>0$ as the side of the square, so $a=3$; (b) when applying Pythagoras Theorem calculate the length of the polygon diagonal, when the lengths of the legs are known as $a=3$ and $b=4$

$c^2 = a^2 + b^2$ $c^2 = 25$ $c= -5$ or $c=5$ because $c>0$ as the length of the leg, so $c=5$;

(M5) use consistently and with full awareness the theorem: for all real numbers $\sqrt{x^2} = |x|$;

(M6) interpret the solution of the same equation in various number sets e.g. $x^2=2$ does not have a solution in the set of natural numbers, it does not have a solution in the set of whole numbers and in the set of rational numbers but it does have a solution in the set of real numbers. Emphasize that the equation $x^2=2$ does not have a solution in the number sets we know, e.g. in the set of rational numbers (it does not suffice to say that there is no solution);

(M7) teach solving equations of the type $x^2=4$ by application of theorems about equivalent equations $x^2-4=0$ that is $(x-2)(x+2) =0$ hence $x-2=0$ or $x+2=0$ and finally $x=2$ or $x=-2$;

(M8) solve riddles with students in which each real number is the solution, e.g.: (a) what number should be written into the square in order to obtain the equation □+0=□–0 or □•0=0:□.; (b) think of a number, add the same number to it, add 10, divide the sum by 2, subtract the number you thought of originally, you obtained 5 in the solution. What number did you have in mind? (c) solve riddles with students in which they guess the rule when knowing two number sequences, just like in "number machines:" student 1 says numbers that 'enter': 7, (subsequently 10 then 14), student 2 in turn says numbers that 'exit:' 50, (subsequently 101 and 197). Students give the numbers until the other students guess the rule. In this example: student 1 says any number (x), student 2 gives the square number to the number chosen by student 1 and adds number 1. The rule is as follows: any number that is made a square number and increased by one. (When x is a number that 'enters' then the number that 'exits' is $y=x^2+1$);

(M9) "Create equations" – (apply a reverse procedure to solving equations) students themselves have to create equations, with a set of solutions given, e.g. number 4. Students give examples of such equations: $x+2=6$; $x-4=0$; $x:4 =1$; (during the first stage they write equations and check if number 4 is the solution) only afterwards they realize that when applying theorems about equivalent equations they are able to write many numerous equations which have the same set of solutions,
e.g.: $2x=8$; $2x+1=9$; $3x+1= 9+x$, (Legutko et al., 1991).

In the situation described in example 1, the teacher could have applied the above-mentioned means, perhaps except M2 and M8.

4.4 Example 2

In class during a mathematics lesson fourteen-year-old students had to solve the problem:

A rectangle whose one side is twice as long as the other side has perimeter that equals 30 cm. Calculate the area of the rectangle.

Students began solving the problem individually. A student drew in his notebook a rectangle:

Below he wrote the formulas: *2a + 2b; a·b* and waited, as if he finished the work.

Observer: Why aren't you carrying on, when you started off well?

Student: Because when we solve problems with rectangles we draw a frame and we use two such formulas but I never know when to use which.

Errors revealed in the student's honest answer can be considered mathematical; they refer to the misapprehension of such basic concepts as: rectangle (as a geometric figure), the area of a figure and the perimeter of a figure, as well as the misapprehension of mathematical language: the convention of the graphic representation of geometric figures and algebraic expressions (formulas for area and perimeter). The second part of the student's statement "we use two such formulas but I never know when to use which" reveals also didactic errors ensuing from teaching: eagerness to generalize methods (procedures) of calculating the area in the algebraic format with the application of formulas without an illustration when and how to apply these formulas.

4.5. Example 2 – analysis

Hypothetical causes (H) of these errors**:**

(H1) A misapprehension of the conventions of graphic representations of geometric figures – rectangle is customarily drawn as a broken ordinary line consisting of four segments, with two sides parallel but we conceptualize it as a figure with its inside. In the student's language it is a 'frame' – empty inside. The student describes what he sees in the figure. One may say with a large dose of certainty that in lower grades (4-5 years earlier) students cut rectangles out of a sheet of paper, and painted over the inside of the rectangle; however, in higher grades students encounter both in text books and in classroom such rectangle figures as the one he drew in his notebook. In his statement the student revealed his conflict between an abstract concept of the figure and its graphic representation.

(H2) The student's other conflict refers to the area of a figure: how to calculate the area of the rectangle when the 'frame' is 'empty inside'? The perimeter and area of the rectangle are numbers which are obtained after inserting numbers into the formula. But what do these numbers mean for the student? Most likely, he does not associate the length measurement with the number of length units in case of the perimeter; nor "filling the figure with square units" with the number of area units in case of the area.

(H3) "Confusing the concepts of the perimeter and area of a figure." If students do not have a well formed concept of the perimeter and area of a figure, it often happens that they calculate the perimeter when they are supposed to calculate the area, and they calculate the area when expected to calculate the perimeter. Errors of such type are made by several percent of students (aged 12-16) and even more often by weaker students (Legutko, 2006). Similar percentage of students in the "carpenter" problem (PISA 2003) calculated the area of the flower bed in order to estimate the length of boards needed to fence a flower bed.

(H4) The student's schematic approach to formulas in learning mathematics "because when we solve problems with rectangles we use two of such formulas." The student did not mark on the figure the letters symbolizing the side lengths nor did he write what the letters meant next to the formulas. When teaching about the area of figures, teachers usually demonstrate how to mark the rectangle area and prove the formulas for the area of triangle, parallelogram and trapezoid. However, teachers move on to exercises in formula application too fast. In mathematics education (in Poland) too little emphasis is put on the formula comprehension: what they help us calculate, if we can calculate the area and perimeter of a figure without a formula, what given letters in the formula mean, how to determine the unit in the solution on the basis of a formula, and how to evaluate the solution correctness.

(H5) Systematic approach to solving problems, which sometimes is formulated by students themselves in a form of good advice in the following way: "draw a figure, write formulas, transform them, substitute data, calculate and you get the result."

4.6. Example 2 – suggested corrective means

How to teach geometric figures, their area and perimeter?

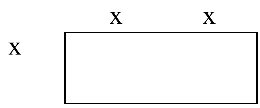
(M1) Discuss with students the role of graphic representation in mathematics teaching. A graphic representation is to facilitate our imagination, to contain essential characteristics of the concept represented according to a certain agreement, e.g., equal segments are equal in the graph, parallel segments are parallel in the graph, and right angles in the plane figure are right angles in the graph.

(M2) Form the concept of a geometric figure. Clearly distinguish real objects with a shape of geometric figure, the graph of a figure and geometric figure as abstract mathematical concepts. When reviewing the figure characteristics do not be satisfied with students' statements: "rectangle has four sides, opposite sides are parallel and equal, and has four right angles." In accepting such constitutive features of a rectangle, we sometimes forget that it is a plane figure and it has the inside.

(M3) Form the concept image of the area of a figure as a result of measuring the figure with area units. Students should associate the area of a plane figure with the land area, (e.g. a ploughed plot, mowed lot or a painted surface) and with defining (estimating) its size with the use of area units. Perhaps it is necessary that the situations connected with the area should be paradigmatic examples for students. Students should have area units "in their eyes," as it were, – cm^2 as a square with a side length 1cm; dm^2 as a square with side length 1dm; m^2 as a square with a side length 1m, are as a square with a side length 10m and so on. The basis for conceptualization of the figure area concept are empirical experiments with measuring the figure; a selection of an area unit, filling the figure with a chosen unit, counting the units and giving the result – the number of area units together with the area unit. Students realize in the empirical experiments that the result is given approximately with a certain margin of error. Useful methods to calculate the units are expressed by means of formulas: therefore, the area of the rectangle is the number of unit squares equal to the product of rectangle side lengths with a common apex, measured with the same unit as the area unit. This individual, empirical experience of students cannot be substituted by teachers' description of how the measurement process is carried out or by the teachers' analysis of the process on a graph.

(M4) Are the formulas indispensable for calculating the perimeter of polygons? I think not. It suffices to comprehend and to often refer to the concept of the perimeter of the polygon – that it is the sum of the lengths of all the sides (and to rationally carry out the calculations). Memorizing (without comprehension) “the formula for the perimeter of the rectangle” might be evaluated as a superfluous didactic action. However, when introducing students to the early stages of algebra: teachers very often (in Polish education) illustrate on the example of the perimeter of the rectangle operations on expressions and the applications of the distributivity of multiplication over addition $a+b+a+b = a+a+b+b= 2a+2b=2(a+b)$ and the product of the expressions a and b by references to the area of the rectangle. Problems in introducing students to the early stages of algebra: the usage of letters to name objects and teaching students to conduct operations on algebraic expressions is an important and difficult task in mathematics teaching. I am not going to discuss this issue and errors associated with it here; however, teachers need to be aware that students’ difficulties with formulas in calculating the area and the perimeter of figures might be rooted in their comprehension of algebraic expressions.

(M5) The above-mentioned observed student revealed an active approach to the problem: he made an effort, manifested a willingness to solve the problem, copied the actions which are most often undertaken in such situations, wrote down the necessary formulas and waited. What for? He waited either for teacher’s help or for some other student, not necessarily him, to present the solution on the board. What was missing in his attitude? A rational approach to the problem. If he asked and answered his own question, “what do I need to know in order to carry on the instruction in the problem?” then he might have first looked for the length of one side, for example like this:



if he marked a shorter side with x, the longer side with $2x$, then the length of all sides would be $6x$ and he could easily calculate the length of the shorter side and then the longer side from the equation $6x =30$. The application of traditionally used letters a and b to mark the lengths of the sides and to write down that $b=2a$, and then the perimeter $2a + 2b = 2a + 2(2a) = 2a + 4a$ would require more advanced skills in the transformation of algebraic expressions.

In the class situation described in example 2, another student suggested a solution to the problem. Had the observer not asked the student, the entire incident would have most likely been overlooked by the teacher.

In this case it was hard to expect a quick brief reaction from the teacher; perhaps the concretization S1 (what is “the drawn frame” to a rectangle?) and S5 (write down what you know about the lengths of the sides in the given rectangle) would have helped the student to move on with the problem solution. Would it clarify for the student his difficulties associated with geometric figures and their area? I doubt it.

5. CONCLUSIONS FOR TEACHING-RESEARCH

Students have the right to err. An error might teach a lot both students and teachers if it evokes a reflection. Teachers cannot be afraid of students’ errors. Teachers should try to understand students’ errors; sometimes students’ errors reveal more than a correct answer. A conversation with students often clarifies more than a long analysis of their written creations and a long search for the error origins.

Errors have to be corrected methodologically: (a) try to make students aware of their errors; and create such situations in which students can discover their errors themselves. First ask: "*why so?*" ("Look at what you did. How did you get there? Are you sure of that?") (b) If questions are not helpful, then the next action teachers should take is to *create contradictions*, *contrasts* or *to give a counterexample*; (c) if students do not correct their errors themselves, teachers can use the *help of other students*. An error analysis and correction with the participation of good students can be educational for others, also for good students (Freudenthal, 1989), and (d) sometimes it is possible (or even necessary) to postpone the error discussion for the next class. It is important that students correct their errors themselves. Errors can not play their role when they are quickly corrected by teachers or other students.

It is better to prevent errors than to correct them (even methodologically). Teachers should not save time on forming the concepts in mathematics teaching and should control the concept comprehension particularly in the early stages and clarify the arising ambiguities and difficulties. When forming the skills in the first stages, teachers should control (without grading) if all simple actions are carried out properly and in an adequate order. It is exactly this moment in the teaching process when the space is created for what Krygowska (1988a) calls "an error provocation" – creating such a situation in which students' errors will be revealed and an opportunity will be created to clarify them.

It is significant for the error prevention to systematically teach students self-control (auto-control). It requires knowledge, skills, patience and understanding on the part of teachers.

REFERENCES

Booker, G. (1988). "The Role of Errors in the Construction of Mathematical Knowlegde." *The Role Errors Play in the Learning and Teaching of Mathematics.* CIEAEM 39. University of Sherbrooke (Canada). 63-69. ["Rola błędów w konstrukcji matematycznej wiedzy." *Roczniki PTM Dydaktyka Matematyki* (1989). 11, 99-108.]

Ćwik, M. (1987). "Reakcja nauczyciela na błąd popełniony przez ucznia w procesie tzw. kontroli ciągłej." *Problemy kształcenia i wychowania*. IKN, Nowy Sącz. 72-94.

Dybiec, Z. (1996). *Błędy w procesie uczenia matematyki (Próba syntezy).* Kraków: Uniwersytet Jagielloński.

Freudenthal, H. (1988). "Erreurs du professeur-analyse didactique de soi-meme." *The Role Errors Play in the Learning and Teaching of Mathematics.* CIEAEM 39. 37-41. ["Błędy nauczyciela-analiza dydaktyczna samego siebie." *Roczniki PTM Dydaktyka Matematyki.*(1989). 11, 109-115.]

Krygowska, A. Z. (1977). *Zarys dydaktyki matematyki.* v.1. Warszawa:WSiP.

--- (1988). "A Study of Certain Types of Errors Linked to the Symbolic Terminology of Elementary Algebra". *The Role Errors Play in the Learning and Teaching of Mathematics.* CIEAEM 39. University of Sherbrooke (Canada). 291-299.

--- (1988a). "Comprendre l'erreur en Mathematiques", CIEAEM 39, 12-16. ["Zrozumieć błąd ucznia." *Roczniki PTM Dydaktyka Matematyki.*(1989). 10, 141-147.]

Legutko M., Stańdo, J. (2006). "Odróżnić pole figury od obwodu - problem, czy brak wiedzy?" *Materiały konferencyjne XX Szkoły Dydaktyków Matematyki*. Bielsko Biała.

Legutko M., Legutko M., Turnau S. (1991). *Matematyka I, podręcznik dla liceum ogólnokształcącego i liceum zawodowego.* Warszawa: WSiP.

Migoń, J. (2007). "Diagnostic Procedure in Algebra Course of the First Grade of the Middle School." *Mathematics Teaching-Research Journal On-Line*. 2, March 5, 2007.

Pellerey, M. (1988). "Pour un cadre de reference du theme." *The Role Errors Play in the Learning and Teaching of Mathematics.* CIEAEM 39. University of Sherbrooke (Canada). 5-11. ["Wprowadzenie do tematu: rola błędu w uczeniu się i nauczaniu matematyki." *Roczniki PTM Dydaktyka Matematyki.*(1989). 10, 133-140.]

Polya, G. (1945). *How to Solve It.* Princeton University Press.

Rouche, N. (1988). "Questions sur les erreurs." *The Role Errors Play in the Learning and Teaching of Mathematics.* CIEAEM 39. Canada: University of Sherbrooke. 97-121. ["Problemy dotyczące błędów." *Roczniki PTM Dydaktyka Matematyki*. (1989). 11, 133-163.]

Thom, R. (1974). "Matematyka 'nowoczesna:' pomyłka pedagogiczna i filozoficzna? Czy istnieje matematyka nowoczesna?" *Wiadomości Matematyczne. XVIII*,113-142.

Turnau S. (1990). *Wykłady o nauczaniu matematyki.* Warszawa: PWN.